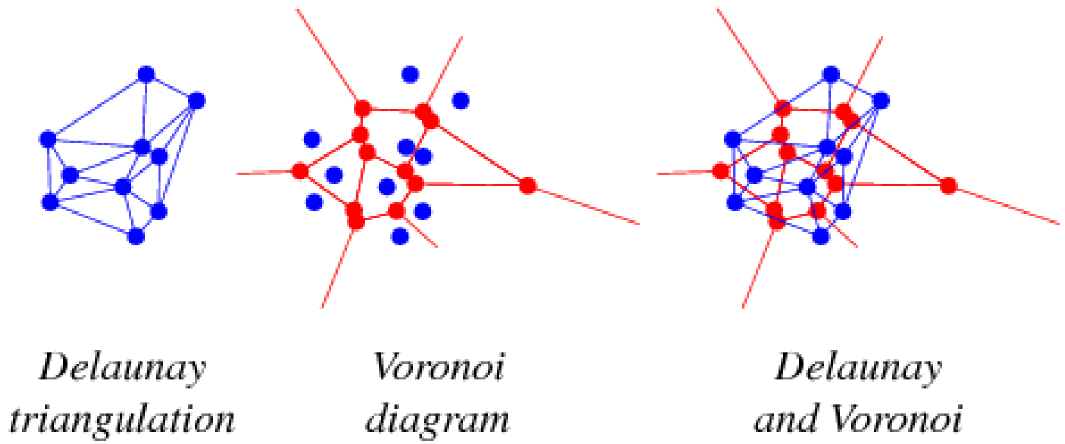
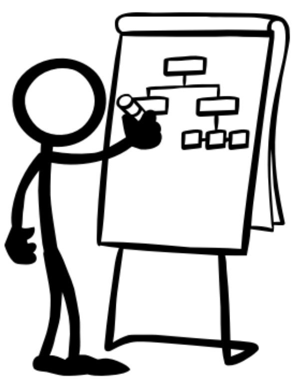
Voronoi Diagram

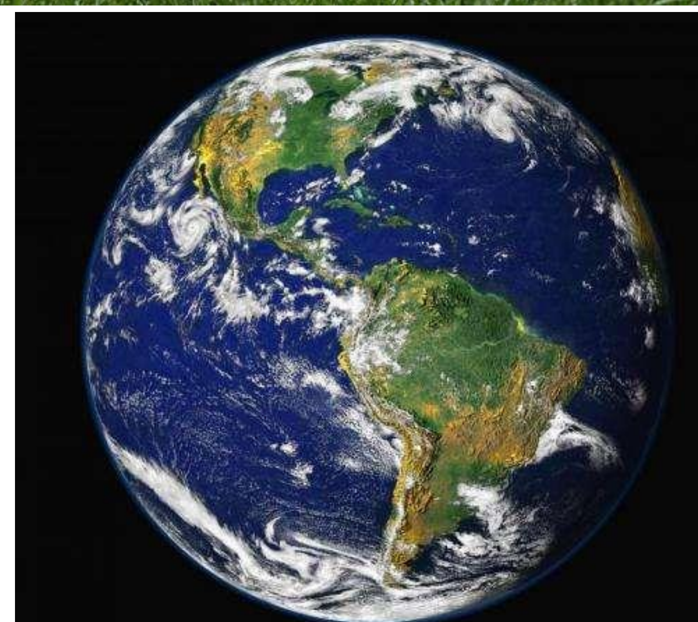




ZHENG Yufei
郑羽霏
יופיי ז'נג

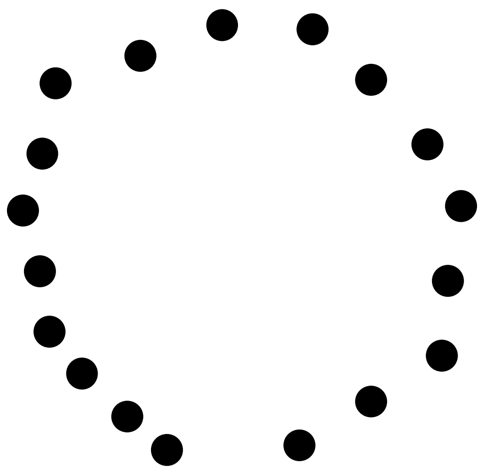
Jan. 17, 2017

How round is an object?



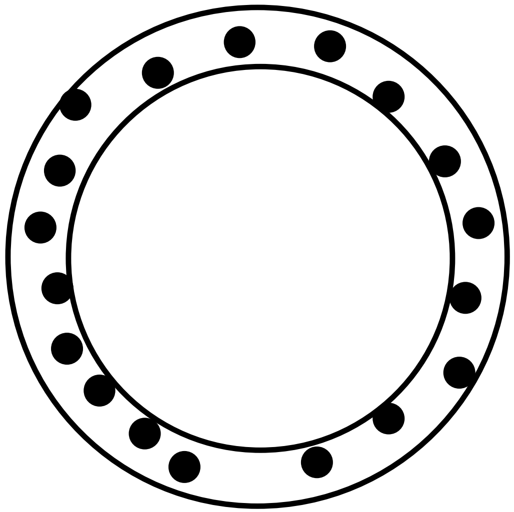
How round is an object?

- Formal problem:
- Given samples from the surface of a quasi-circular object, we would like to quantify how round it is.



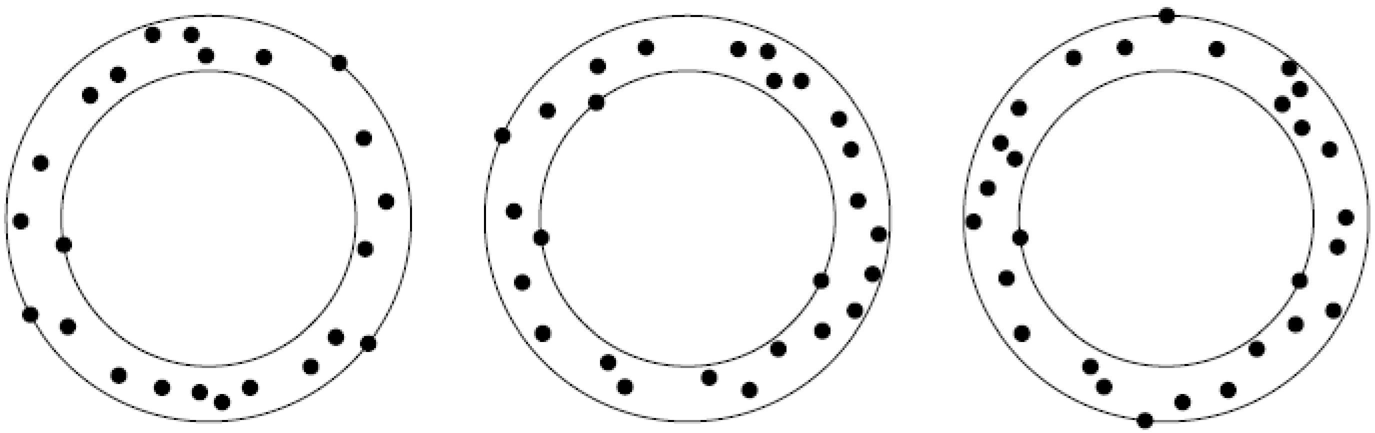
Smallest width ring

- We can come up with many measures
- We will consider the following measure: What is the width of the minimal ring that contain all the samples?



Smallest width ring

- Observations:
- It suffice to find the center of the ring
- The rings are determined by 4 points



Case 1:
3 outer 1 inner

Case 2:
1 outer 3 inner

Case 3:
2 outer 2 inner

Ordinary Voronoi Diagram - Recall

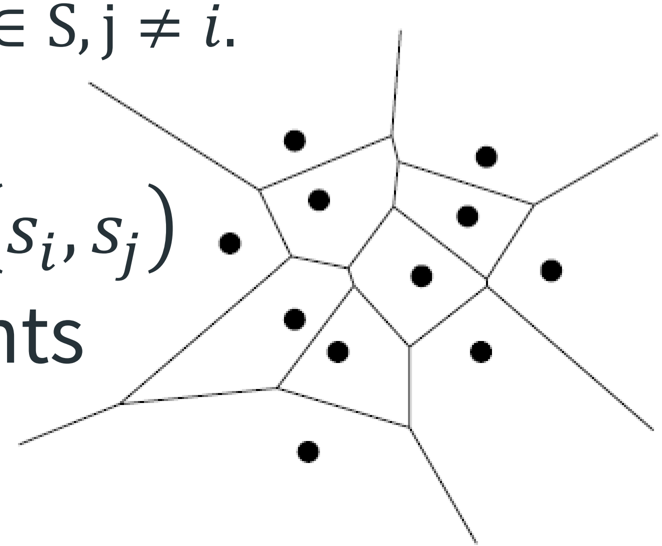
- **Definition** – a subdivision of plane into cells
  - Sites: $S = \{s_1, s_2, \cdots, s_n\}$
  - Euclidean distance in the plane

$$\mathrm{dist}(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}.$$

  - p lies in the cell of site s_i iff

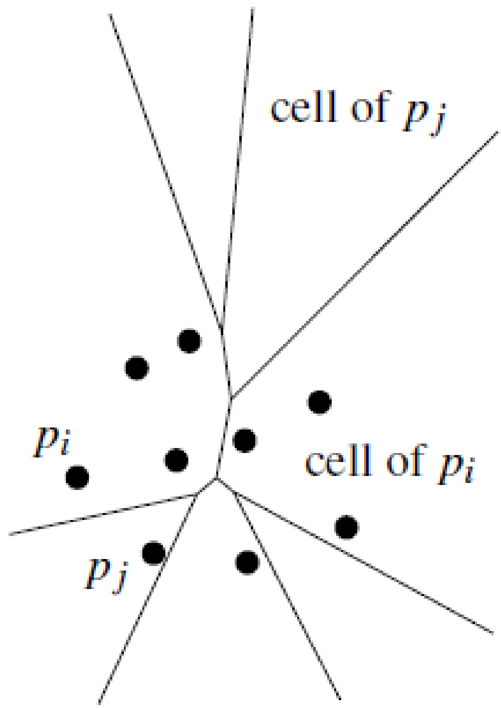
$$\mathrm{dist}(p, s_i) < \mathrm{dist}(p, s_j), \forall s_j \in \mathrm{S}, \mathrm{j} \neq i.$$

- Cells - $V(s_i) = \cap_{1 \le j \le n, j \ne i}\, h(s_i, s_j)$
- Edges - straight line segments



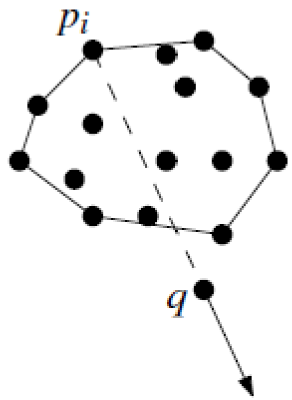
Farthest point Voronoi diagram

- Each cell is associated with the **farthest** point from the cell



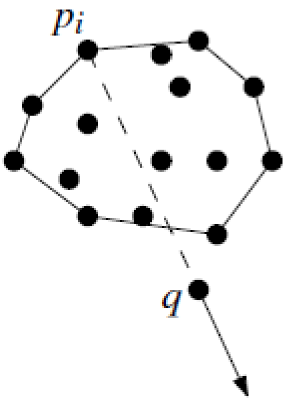
Farthest point Voronoi diagram

- Observations:
- The diagram is the intersection of the "Other side" of the bisector half-planes.
- A point p has a cell iff p is a vertex of the convex hull of the point.
- If the farthest point from q is p_i, then, the ray from q in the opposite direction to p_i is also in the cell of p_i.
  - ⇒ The cells are unbounded.
- The separator between the cells of p_i and p_j is the bisector of p_i and p_j



Farthest point Voronoi diagram

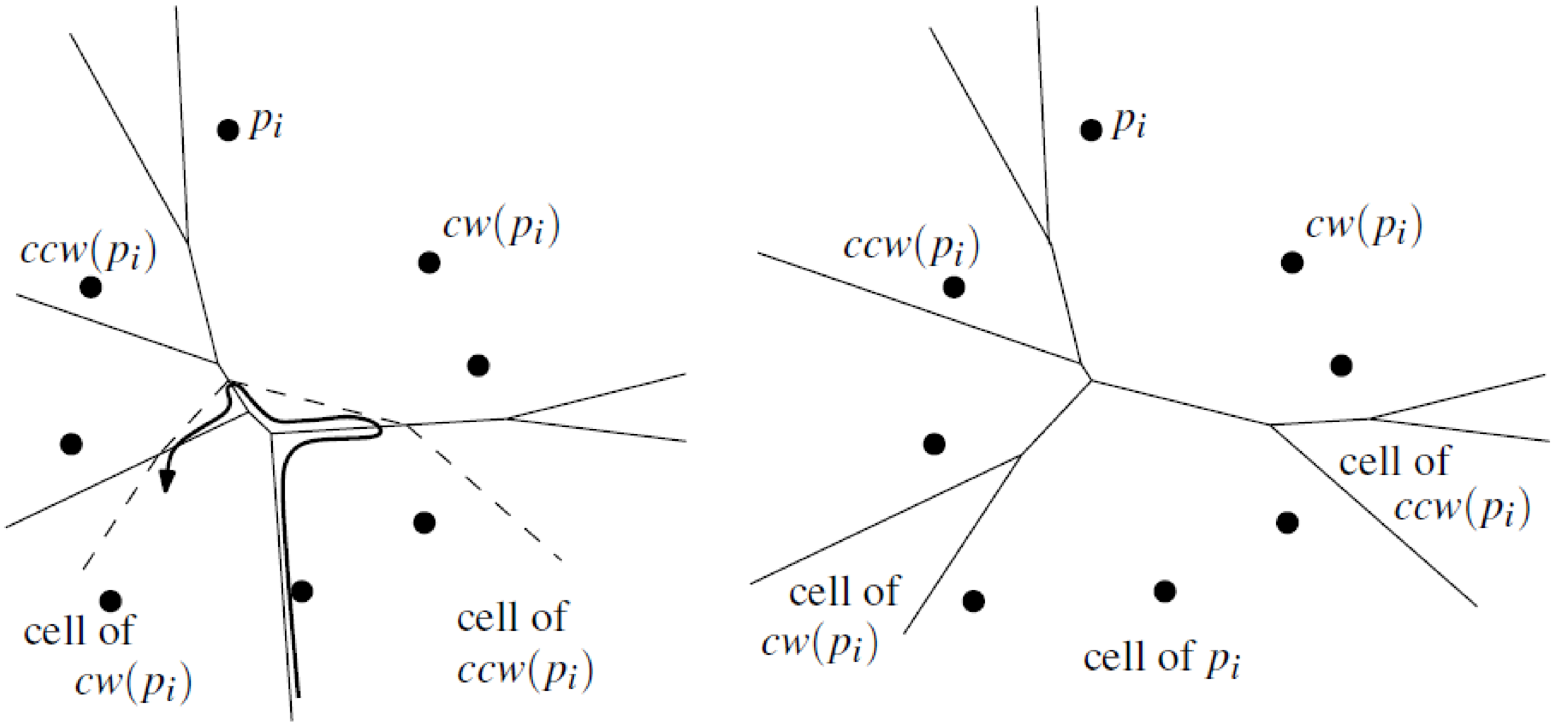
- Observations:
- The diagram is the intersection of the “Other side” of the bisector half-planes.
- A point p has a cell iff p is a vertex of the convex hull of the point.
- If the farthest point from q is p_i, then, the ray from q in the opposite direction to p_i is also in the cell of p_i.
  - ⇒ The cells are unbounded.
- The separator between the cells of p_i and p_j is the bisector of p_i and p_j



Farthest point Voronoi diagram

- Consider a random order of the CH vertices, $p_1, \dots, p_h$
- Given a diagram for $p_1, \dots, p_{i-1}$ we would like to add p_i
- We will denote the neighbors of p_i (when p_i is added) as $cw(p_i)$ and $ccw(p_i)$
- How do we find $cw(p_i)$ and $ccw(p_i)$?
  - Remove the points in the opposite order, the neighbors when p_i is removed are $cw(p_i)$ and $ccw(p_i)$

Farthest point Voronoi diagram

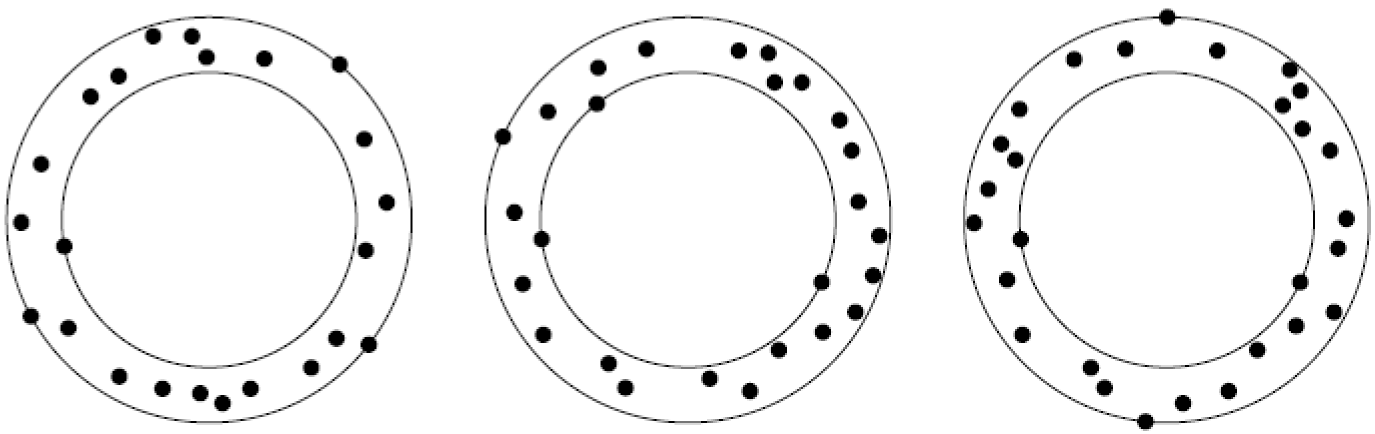


Farthest point Voronoi diagram

- Complexity:
- CH - $O(n \log n)$
- Insertion of p_i: worst case O(i)
  Expected: O(1)
- Proof:
- The complexity of the ith insertion is as the complexity of the cell of p_i
- There are at most $2i - 3$ edges after the ith insertion
- $\Rightarrow$ The average cell complexity is $O(1)$
- Each point from $p_1, \ldots, p_i$ have the same probability to be the last one added $\Rightarrow$ the expected complexity of insertion is $O(1)$
- Corollary: the expected complexity is $O(n \log n)$
  and the worst case complexity is $O(n^2)$.

Back to the smallest width ring

- Case 1: the center is a vertex of the farthest point Voronoi diagram
- Case 2: the center is a vertex of the closest point Voronoi diagram
- Case 3: the center is an intersection of two edges from both diagrams.



Case 1:
3 outer 1 inner

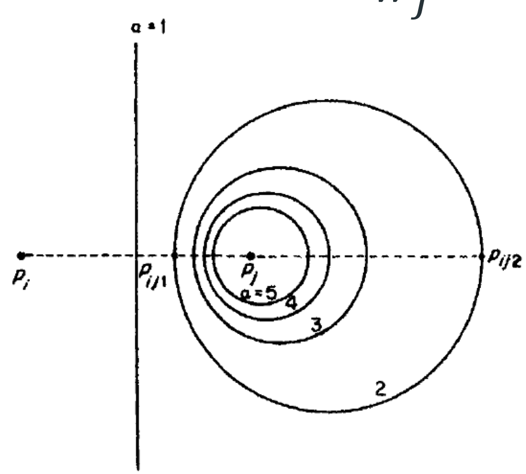
Case 2:
1 outer 3 inner

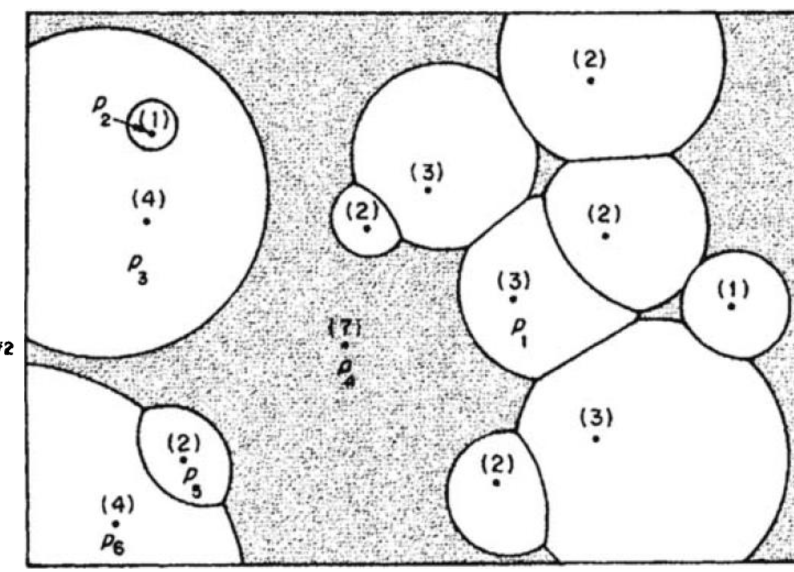
Case 3:
2 outer 2 inner

Multiplicatively Weighted Voronoi Diagram

- **Difference** – Euclidean distance between points is divided by positive weights
  - **Distance** - $\text{dist}(p, s_i) = \frac{\|p - s_i\|}{w_i}$.
- Edges – circular arcs or straight line segments
  - For every point x on the edge separating $V(s_i)$ and $V(s_j)$,

$$\text{dist}(x, s_i) = \text{dist}(x, s_j) \cdot \frac{w_i}{w_j}.$$



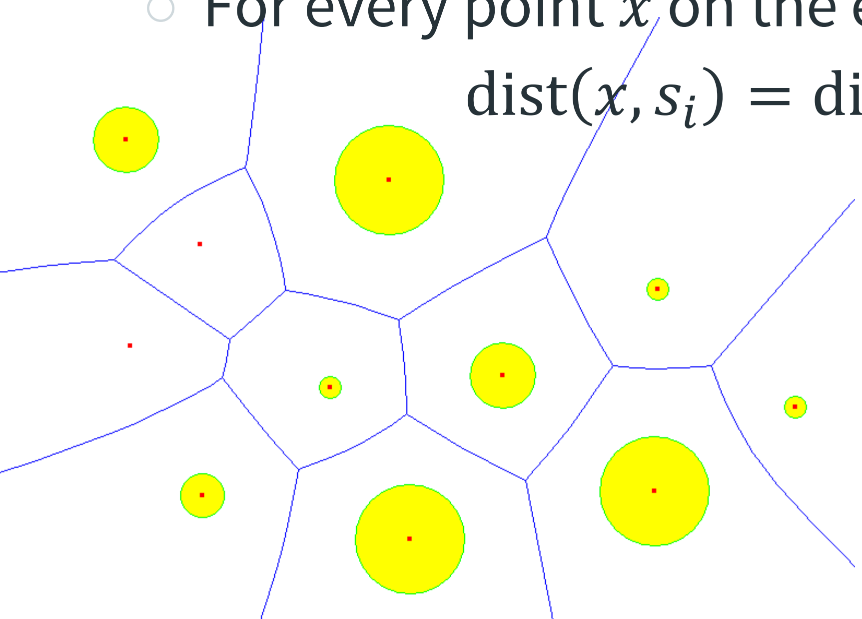


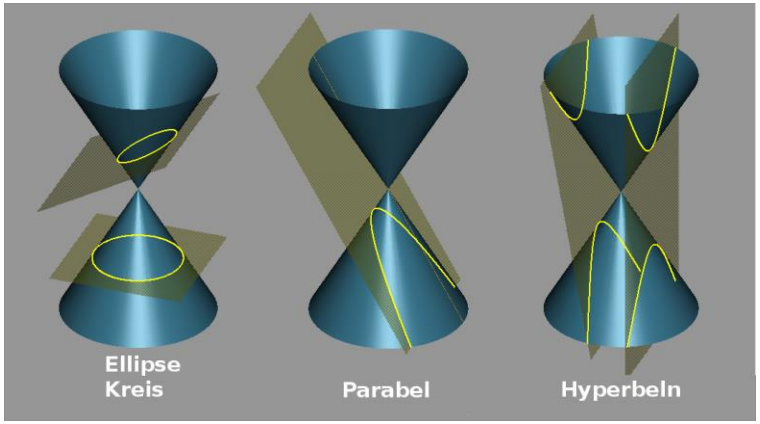
Additively Weighted Voronoi Diagram

- **Difference** – positive weights are subtracted from the Euclidean distance
  - **Distance** - $\operatorname{dist}(p, s_i) = \|p - s_i\| - w_i$.

- Edges – hyperbolic arcs or straight line segments
  - For every point x on the edge separating $V(s_i)$ and $V(s_j)$,

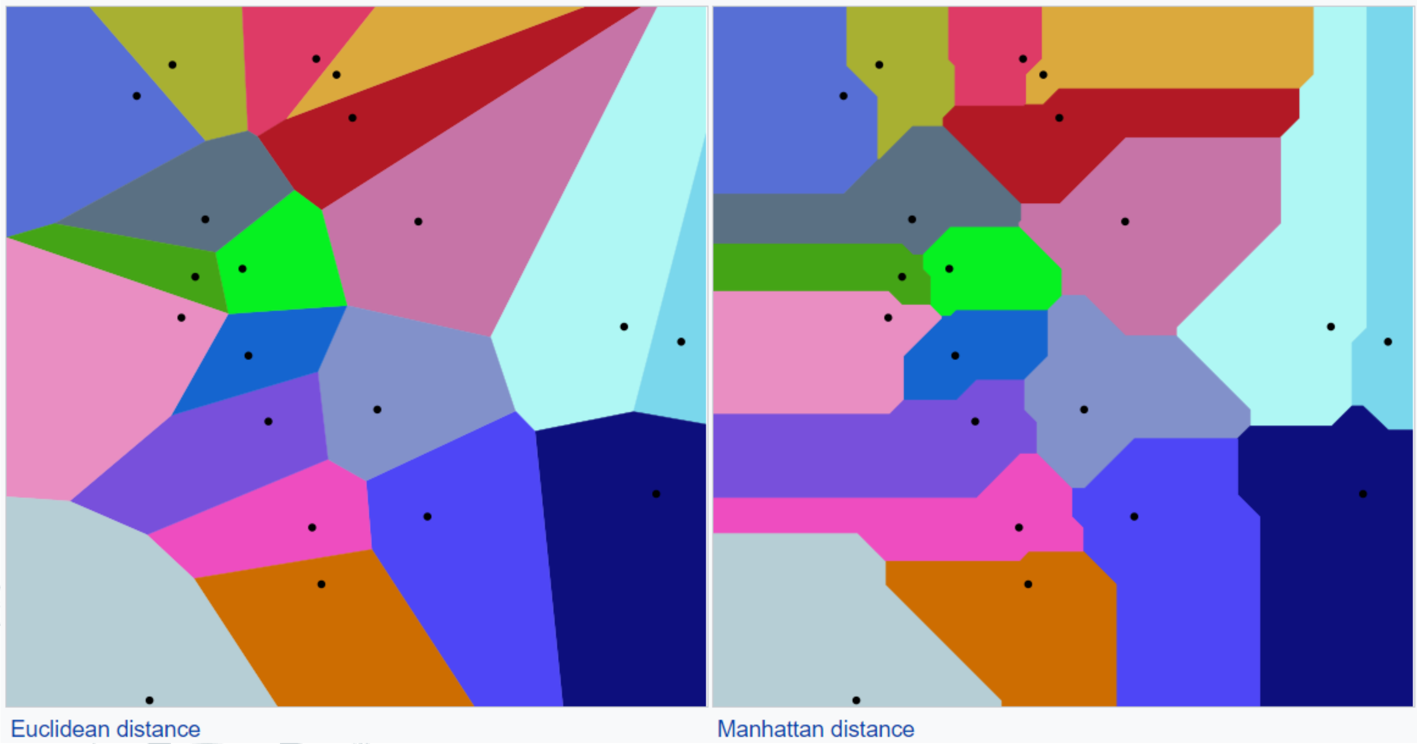
$$\operatorname{dist}(x, s_i) = \operatorname{dist}(x, s_j) + (w_i - w_j).$$





Voronoi Diagram in Different Metric

- **Difference** – Distance defined in L_1
  - **Distance** - $\text{dist}(p, s_i) = |p_x - s_{i,x}| + |p_y - s_{i,y}|$.
- Edges – vertical, horizontal or diagonal at ± 45 degree



Centroidal Voronoi Diagram (CVD)

- **Difference** – Each site is the mass centroid of each cell
  - Given a region $V \in \mathrm{R}^N$, and a density function ρ,
    mass centroid $\boldsymbol{z}^*$ of V is defined by $\boldsymbol{z}^* = \dfrac{\int_V \boldsymbol{y}\rho(\boldsymbol{y})\, d\boldsymbol{y}}{\int_V \rho(\boldsymbol{y})\, d\boldsymbol{y}}$
  - **Centroid of polygon** (CCW order of the vertices (x_i, y_i))

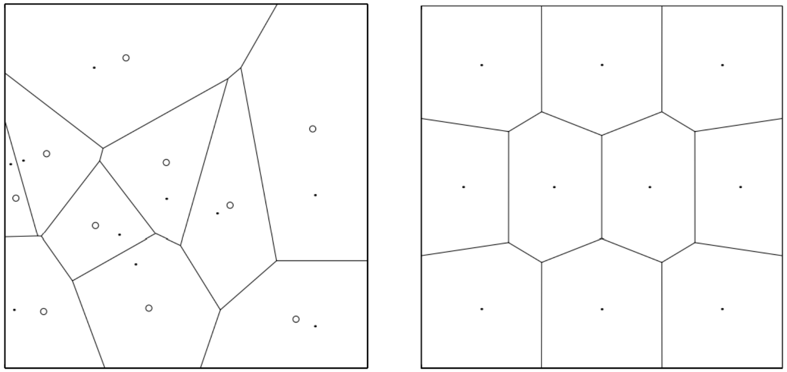
$$Area = A = \frac{1}{2}\sum_{i=0}^{N-1}(x_i y_{i+1} - x_{i+1} y_i)$$

$$x_c = \frac{1}{6A}\sum_{i=0}^{N-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$y_c = \frac{1}{6A}\sum_{i=0}^{N-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

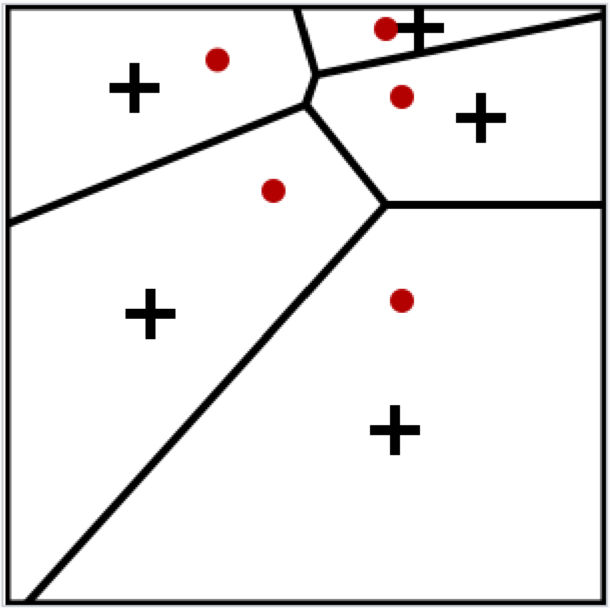
CVD Computation – Lloyd's Algorithm

1. Compute the Voronoi Diagram of the given set of sites $\{s_i\}_{i=1}^{n}$;
2. Compute the mass centroids of Voronoi cells $\{V_i\}_{i=1}^{n}$ found in step 1, these centroids are the new set of sites;
3. If this new set of sites meets the **convergence criterion**, terminate;
   Else, return to step 1.

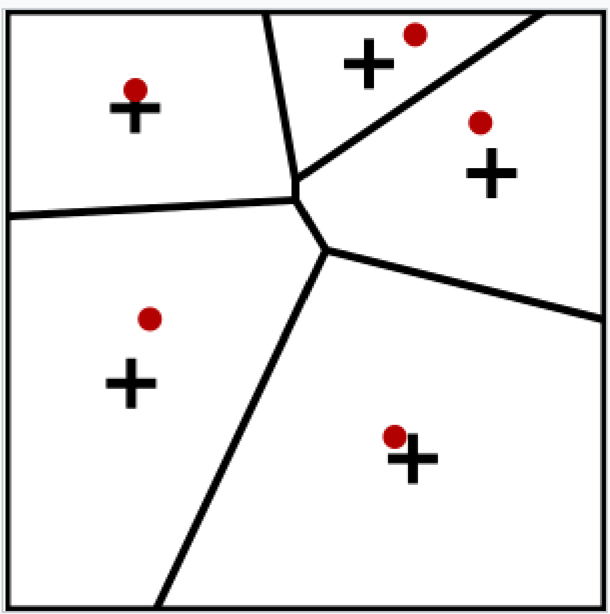


Note

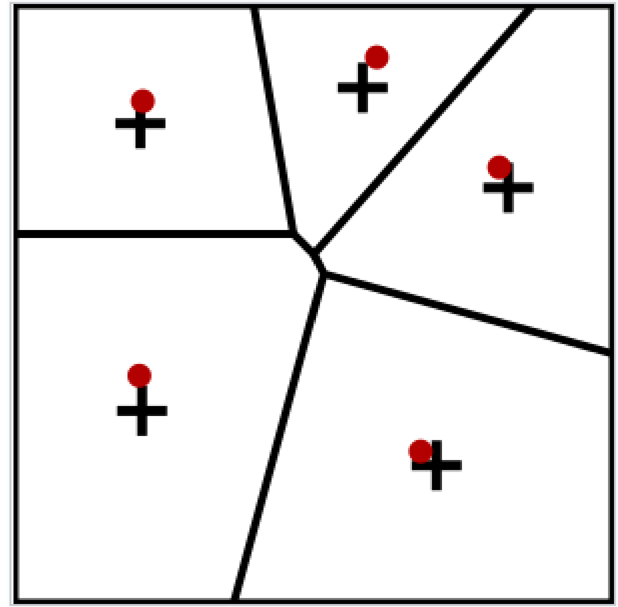
- Convergence criterion depends on specific application
- Converges to a CVD slowly, so the algorithm stops at a tolerance value
- Simple to apply and implement



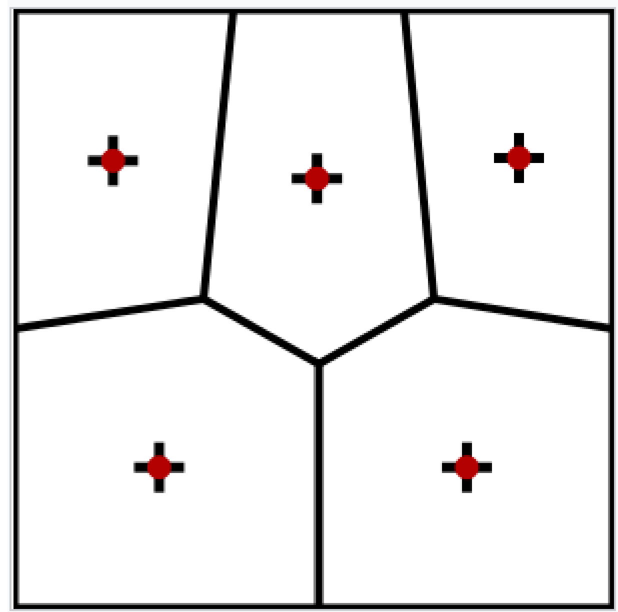
First iteration



Second iteration



Third iteration



Fifteenth iteration

Voronoi Diagram in Higher Dimensions

- **Cells** – convex polytopes
- **Bisectors** - $(d-1)$-dimensional hyper-planes
- **Complexity** - $O(n^{\lceil \frac{d}{2} \rceil})$